Excitation of Betatron Oscillation under Finite Chromaticity

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1 Introduction

The betatron oscillation of a beam in a storage ring with finite chromaticity, excited by external shaking force, is analyzed. This can be applied to to the energy spread measurement of a beam.

2 Single Particle Motion

2.1 *Green Function*

We assume that energy oscillation of an electron executing synchrotron motion is

$$\delta = \hat{\delta}\cos(\omega_s t + \phi) \tag{1}$$

where $\delta = \frac{E-E_0}{E}$ is a relative energy shift from the reference energy E_0 of the ring and ω_s is a synchrotron frequency.

With finite chromaticity ξ, betatron frequency has time dependence as

$$\omega(t) = \omega_\beta\left[1 + \xi\hat{\delta}\cos(\omega_s t + \phi)\right] \tag{2}$$

We define oscillating part of the tune as $\hat{\Delta\omega} = \omega_\beta\xi\hat{\delta}$ for later discussion.

The equations of betatron oscillation of an electron under chromaticity, excited by a external force f, is

$$\frac{dq}{dt} = p \tag{3}$$

$$\frac{dp}{dt} = -\omega(t)^2 q - 2\alpha p + F(t) \tag{4}$$

$$F(t) = \frac{f}{m\gamma} \tag{5}$$

where q is a transverse coordinate and γ is the Lorentz factor of electrons.

We will solve above equations under the assumption that

- small damping rate; $\alpha << \omega(t)$
- small modulation amplitude; $\frac{\hat{\Delta\omega}}{\omega} << 1$
- synchrotron frequency is slow compared to betatron frequency; $\frac{\omega_s}{\omega} << 1$.

The Green function of the system is the solution for q with $F(t) = \delta(t - t')$ and is

$$\begin{aligned} G(t,t') &= -i\frac{1}{\sqrt{\omega(t)\omega(t')}} \\ &\times e^{-\alpha(t-t')+i\int_{t'}^{t}\omega'(t'')dt''} \quad (6)\\ &= -i\frac{1}{\sqrt{\omega(t)}}e^{-\alpha t+i\int_0^t \omega'(t'')dt''} \\ &\times e^{i\frac{\hat{\Delta\omega}}{\omega_s}\sin\phi}\sum_{n=-\infty}^{\infty} J_n\left(\frac{\hat{\Delta\omega}}{\omega_s}\right)e^{-in\phi} \\ &\times \frac{1}{\sqrt{\omega(t')}}e^{\left(\alpha - i(\omega_0' + n\omega_s)\right)t'} \quad (7)\end{aligned}$$

where, from the assumptions, we use

$$\omega'(t) = \sqrt{\omega(t)^2 - \alpha^2} \simeq \omega(t) \tag{8}$$

$$\int_0^{t'}\omega'(t'')dt'' = \omega_\beta t + \frac{\hat{\Delta\omega}}{\omega_s}\left[\sin(\omega_s t + \phi) - \sin\phi\right] \tag{9}$$

and the definition of Bessel functions;

$$e^{ix\sin\phi} = \sum_{n=-\infty}^{n=\infty} J_n(x)e^{in\phi}. \tag{10}$$

This form of the solution can be anticipated from the fact that the system is adiabatic for change of the betatron frequency and the action is a adiabatic invariant.

2.2 *Excitation by Shaker*

We assume that one shaker in a ring excites betatron oscillation of a beam and length of the shaker is short enough compared to the value of beta function at the location of the shaker. With this assumption, we can use δ-function to represents the shaker force then the external force in Eq. (4) is

$$\begin{aligned} F(t) &= c\hat{\theta}\cos(\omega_f t + \psi)\sum_{k=-\infty}^{k=\infty}\delta(t - kT_0) \\ &= \frac{c\hat{\theta}}{2}\left(e^{i(\omega_f kT_0+\psi)} + e^{-i(\omega_f kT_0+\psi)}\right) \\ &\times \sum_{k=-\infty}^{k=\infty}\delta(t - kT_0) \quad (11)\end{aligned}$$

where $\hat{\theta}$ is the kick angle amplitude of shaker force.

The solution for $0 < t < T_0$ can be obtained using the Green function Eq. (7) and is

$$\begin{aligned} q(t) &= \int_{-\infty}^{t} G(t,t')F(t')dt' \quad (12)\\ &= \frac{c\hat{\theta}}{2}(-i)\frac{1}{\sqrt{\omega(t)}}e^{-\alpha t+i\int_0^t\omega'(t)dt'} \\ &\times e^{i\frac{\hat{\Delta\omega}}{\omega_s}\sin\phi}\sum_{n=-\infty}^{\infty} J_n\left(\frac{\hat{\Delta\omega}}{\omega_s}\right)e^{-in\phi}\end{aligned}$$

$$\times \sum_{k=-\infty}^{0} \frac{1}{\sqrt{\omega(kT_0)}} \times \left[e^{(\alpha-i(\omega_\beta+n\omega_s-\omega_f))kT_0+i\psi} + e^{(\alpha-i(\omega_\beta+n\omega_s+\omega_f))kT_0-i\psi}\right] \quad (13)$$

$$= \frac{c\hat{\theta}}{2}(-i)\frac{1}{\omega_\beta}e^{-\alpha t+i\int_0^t \omega'(t)dt'} \times e^{i\frac{\hat{\Delta\omega}}{\omega_s}\sin\phi}\sum_{n=-\infty}^{\infty} J_n\left(\frac{\hat{\Delta\omega}}{\omega_s}\right)e^{-in\phi} \times \left[\frac{e^{i\psi}}{1-e^{(-\alpha+i(\omega_\beta+n\omega_s-\omega_f))T_0}} + \frac{e^{-i\psi}}{1-e^{(-\alpha+i(\omega_\beta+n\omega_s+\omega_f))T_0}}\right] \quad (14)$$

$$= \frac{c\hat{\theta}}{2}(-i)\frac{1}{\omega_\beta}e^{-\alpha t+i\int_0^t \omega'(t)dt'} \times e^{i\frac{\hat{\Delta\omega}}{\omega_s}\sin\phi}\sum_{n=-\infty}^{\infty} J_n\left(\frac{\hat{\Delta\omega}}{\omega_s}\right)e^{-in\phi} \times h(\omega_f) \quad (15)$$

where

$$h(\omega_f) = \frac{e^{i\psi}}{1-e^{(-\alpha+i(\omega_\beta+n\omega_s-\omega_f))T_0}} + \frac{e^{-i\psi}}{1-e^{(-\alpha+i(\omega_\beta+n\omega_s+\omega_f))T_0}} \quad (16)$$

and we uses the assumption that $\omega(t) \simeq \omega_\beta$ to modify Eq. (13) to Eq. (14).

The reference for time is the phases, ϕ and ψ, which appear in Eq. (1) and Eq. (11), respectively, hence you can obtain $q(t)$ for time $nT_0 < t < (n+1)T_0$ by shifting these phase as $\phi \to \phi + \omega_s nT_0$ and $\psi \to \psi + \omega_s nT_0$.

Now we have a motion of an electron excited by a shaker in a ring with chromaticity.

3 Bunch Motion

Eq. (15) is a motion of an electron of the amplitude $\hat{\delta}$ and the initial phase ϕ of energy oscillation. To get a bunch motion, we have to sum it up for all electrons in a bunch using longitudinal distribution function. First, we will expand the former part of Eq. (15) using Eq. (9);

$$q(t) = -i\frac{c\hat{\theta}}{2\omega_\beta}e^{-i\frac{\hat{\Delta\omega}}{\omega_s}\sin\phi}e^{i\frac{\hat{\Delta\omega}}{\omega_s}\sin\phi} \times \sum_{n'=-\infty}^{\infty} e^{(-\alpha+i(\omega_\beta+n'\omega_s)t)}J_{n'}\left(\frac{\hat{\Delta\omega}}{\omega_s}\right)e^{in'\phi} \times \sum_{n=-\infty}^{\infty} J_n\left(\frac{\hat{\Delta\omega}}{\omega_s}\right)e^{-in\phi} \times h(\omega_f) \quad (17)$$

$$= -i\frac{c\hat{\theta}}{2\omega_\beta} \times \sum_{n'=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} e^{(-\alpha+i(\omega_\beta+n'\omega_s))t} \times J_{n'}\left(\frac{\hat{\Delta\omega}}{\omega_s}\right)J_n\left(\frac{\hat{\Delta\omega}}{\omega_s}\right)e^{i(n'-n)\phi} \times h(\omega_f). \quad (18)$$

We assume that the longitudinal distribution of electrons in a bunch is Gaussian,

$$g(\hat{\delta},\phi)\hat{\delta}d\hat{\delta}d\phi = \frac{1}{2\pi\sigma_\delta^2}e^{-\frac{\hat{\delta}^2}{2\sigma_\delta^2}}\hat{\delta}d\hat{\delta}d\phi, \quad (19)$$

then we have the bunch motion

$$\overline{q}(t) = \int_0^{2\pi}d\phi\int_0^\infty \hat{\delta}d\hat{\delta}q(t)g(\hat{\delta},\phi) = -i\frac{c\hat{\theta}}{2\omega_\beta}\sum_{n'=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} e^{(-\alpha+i(\omega_\beta+n'\omega_s))t} \times \int_0^\infty J_{n'}\left(\frac{\hat{\Delta\omega}}{\omega_s}\right)J_n\left(\frac{\hat{\Delta\omega}}{\omega_s}\right)\frac{1}{\sigma_\delta^2}e^{-\frac{\hat{\delta}^2}{2\sigma_\delta^2}}\hat{\delta}d\hat{\delta} \times \frac{1}{2\pi}\int_0^{2\pi}e^{i(n'-n)\phi}d\phi \times h(\omega_f) \quad (20)$$

$$= -i\frac{c\hat{\theta}}{2\omega_\beta}\sum_{n=-\infty}^{\infty} e^{(-\alpha+i(\omega_\beta+n\omega_s))t} \times \int_0^\infty J_n\left(\frac{\hat{\Delta\omega}}{\omega_s}\right)^2\frac{1}{\sigma_\delta^2}e^{-\frac{\hat{\delta}^2}{2\sigma_\delta^2}}\hat{\delta}d\hat{\delta} \times h(\omega_f) \quad (21)$$

where $\hat{\Delta\omega} = \omega_\beta\xi\hat{\delta}$.

Until now, we only treated the case of a ring with constant focusing to simplify the discussion. For actual situation, instead of Eq. (3)-Eq. (4), we have to start with the equations [1]

$$\frac{d\eta}{du} = \zeta \quad (22)$$

$$\frac{d\zeta}{du} = -\nu(t)^2\eta - 2\alpha'\zeta + \nu_0^2\beta^{3/2}\frac{f}{E} \quad (23)$$

where $u = \frac{1}{\nu_0}\int^s \frac{ds'}{\beta} = \frac{1}{\nu_0}\int^t \frac{cdt'}{\beta}$, $\eta = \frac{q}{\sqrt{\beta}}$, $\alpha' = \frac{\nu\beta}{c}\alpha$ and betatron tune $\nu(t) = \nu_0(1+\xi\delta)$. The results is

$$\overline{q}(t) = -i\frac{\hat{\theta}}{2}\sqrt{\beta_o\beta_s}\sum_{n=-\infty}^{\infty} e^{[-\alpha+i(\omega_\beta+n\omega_s)]t} \times \int_0^\infty J_n\left(\frac{\hat{\Delta\omega}}{\omega_s}\right)^2\frac{1}{\sigma_\delta^2}e^{-\frac{\hat{\delta}^2}{2\sigma_\delta^2}}\hat{\delta}d\hat{\delta} \times h(\omega_f) \quad (24)$$

where β_o and β_s are the beta function at the observation point and the beta function at the shaker , respectively, and we set $[-\alpha' + i(\nu_0 + n\nu_s)]u \sim [-\alpha + i(\omega_\beta + n\omega_s)]t$.

4 Amplitude

The function $h(\omega_f)$ has many peaks at $|\omega_f| = |\omega_\beta + n\omega_s|$ where n is integer and the height of these peaks are the same.

If we assume that the distance between peaks is wide enough, the amplitude of betatron oscillation of an electron around $|\omega_f| \sim |\omega_\beta + n\omega_s|$ is

$$|q(t)| = \frac{1}{2}\hat{\theta}\sqrt{\beta_o\beta_s}\left|J_n\left(\frac{\hat{\Delta\omega}}{\omega_s}\right)\right||h(\omega_f)| \qquad (25)$$

and the amplitude of betatron oscillation of a center of mass of a bunch around $|\omega_f| \sim |\omega_\beta + n\omega_s|$ is

$$\begin{aligned}|\overline{q}(t)| &= \frac{1}{2}\hat{\theta}\sqrt{\beta_o\beta_s}\int_0^\infty J_n\left(\frac{\hat{\Delta\omega}}{\omega_s}\right)^2\frac{1}{\sigma_\delta^2}e^{-\frac{\hat{\delta}^2}{2\sigma_\delta^2}}\hat{\delta}d\hat{\delta} \\ &\quad \times|h(\omega_f)|. \qquad (26)\end{aligned}$$

5 Vlasov Equation

For people who familiar with Vlasov equations, It might be easier to get the result for bunch motion through it treating the shaker force as perturbation. If we neglect the damping term, the Vlasov equation for the system Eq. (22)-Eq. (23) is

$$\frac{\partial\psi}{\partial u} + \nu_0(1+\xi\delta)\frac{\partial\psi}{\partial\theta} + \nu_0\beta^{3/2}\frac{f}{E}\sin\theta\frac{\partial\psi}{\partial q} + \nu_s\frac{\partial\psi}{\partial\phi} = 0 \qquad (27)$$

where transverse coordinate is $y = q\cos\theta$. Following usual scheme [2], we can solve above equation by setting

$$\psi = f_0(q)g_0(r) + f_1(q,\theta)g_1(r,\phi)e^{-i\nu u} \qquad (28)$$

$$f_1(q,\theta) = -D\frac{\partial f_0}{\partial q}e^{i\theta} \qquad (29)$$

$$g_1(r,\phi) = \sum_{l=-\infty}^{l=\infty} g_l(r)e^{il\phi}e^{i\frac{\xi\omega_\beta}{\eta c}r\cos\phi}. \qquad (30)$$

where r and ϕ are defined as $z = r\cos\phi$ and $\delta = \frac{\omega_s}{\eta c}r\sin\phi$ using relative distance z and relative energy spread δ. The bunch motion is $<y>= \sqrt{\beta}<\eta>$ and

$$<\eta>= \int d\theta\int qdq\int d\phi\int rdr(q\cos\theta)\psi \qquad (31)$$

, which is the same result as Eq. (24).

6 Energy Spread Measurement

From Eq. (26), we can observe several sideband peaks in the frequency response spectrum of the betatron motion and the peak hight of n-th sideband is proportional to

$$\begin{aligned}I_n\left(y = \frac{\xi\omega_\beta\sigma_\delta}{\omega_s}\right) &= \int_0^\infty J_n\left(\frac{\xi\omega_\beta\hat{\delta}}{\omega_s}\right)^2\frac{1}{\sigma_\delta^2}e^{-\frac{\hat{\delta}^2}{2\sigma_\delta^2}}\hat{\delta}d\hat{\delta} \\ &= \frac{1}{y^2}\int_0^\infty J_n(x)^2e^{-\frac{x^2}{2y^2}}xdx \qquad (32)\end{aligned}$$

because the maximum value of the function $|h(\omega_f)|$ around $\omega_f = \omega \pm n\omega_s$ is independent on n.

This shows that if we measure the ratio of each peaks, we can obtain $\frac{\xi\omega_\beta\sigma_\delta}{\omega_s}$ using Eq. (32) and, if we know ξ and ω_s, we have σ_δ.

7 Conclusion

From Eq. (15), the amplitude of the betatron oscillation of an electron of which amplitude of energy oscillation is $\hat{\delta}$ is reduced by the factor $J_n\left(\frac{\xi\omega_\beta\hat{\delta}}{\omega_s}\right)$ and $\int_0^\infty J_n\left(\frac{\xi\omega_\beta\hat{\delta}}{\omega_s}\right)^2\frac{1}{\sigma_\delta^2}e^{-\frac{\hat{\delta}^2}{2\sigma_\delta^2}}\hat{\delta}d\hat{\delta}$ for a bunch of which the RMS relative energy spread is σ_δ.

We can obtain the energy spread of a beam with this relation measuring the relative peak hight of $n = 0, \pm1, \pm2, ...$ for the beam of small current as collective effects is negligible. The limitation of this method is that this can only be applied to the beam of which bunch current is small enough for collective effects to be negligible.

References

[1] See texts for accelerator physics. For example, S.Y.Lee, "Accelerator Physics," World Scientific,p.108(1999).

[2] A. W. Chao, "Physics of Collective Beam Instabilities in High Energy Accelerators," John Wiley & Sons, p334 (1993)