Stability of 1D Waterbag-type beams

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Abstract

Space charge effects of 1D waterbag-type beams under constant linear focusing force is studied. A self-consistent stationary distribution is obtained in the form of power series, and the linearized Vlasov analysis is applied to investigate the stability of beam plasma oscillation modes.

Summary

Space charge effect is an important subject for accelerator dynamics and beam transport designs and has been studied by many researchers in various ways. In the present paper, we investigate the stability of space-charge-induced plasma oscillation modes by using the linearized Vlasov approach. We treat here 1D beams travelling in uniform focusing channel, and the particle distribution is assumed to be the waterbag-type.

In the following sections, we first try to obtain the stationary distribution in the waterbag model. After that, the usual linearized Vlasov equation is solved together with the Poisson's equation, resulting in an eigen-value problem, *i.e.* Eq.(25) or Eq.(31), which determines the frequencies of the beam plasma oscillation modes. These two eigen-value equations are obtained with the different approximations for the stationary state Hamiltonian. In both cases, the eigen-values are always real, which means that there exists no instability in 1D waterbag-type beams.

The Vlasov-Poisson equation system

1D beam motion under linear external focusing force with space charge field E(x;s) is governed by the equation

$$\frac{d^2 x}{ds^2}+\left(\frac{\omega_0}{u}\right)^2 x-\frac{q}{m_0 u^2}E(x;s)=0 \qquad (1)$$

where u is the beam velocity along the orbit, ω_0 corresponds to the betatron tune at zero current and the independent parameter s=ut. Here, for simplicity, we put the relativistic parameter γ~1 because space charge effects are dominant in low-energy region. Eq.(1) can be derived from the Hamiltonian

$$H=\frac{1}{2}p_x^{\,2}+\frac{1}{2}\left(\frac{\omega_0}{u}\right)^2 x^2+\frac{q}{m_0 u^2}V(x;s) \qquad (2)$$

where the space charge potential V(x;s) satisfies the Poisson's equation

$$\frac{\partial^2 V(x;s)}{\partial x^2}=-\frac{\partial E(x;s)}{\partial x}=-\frac{q}{\varepsilon_0}n(x;s). \qquad (3)$$

In Eq.(3), the particle number density n(x;s) is given by

$$n(x;s)=\int f(x,p_x;s)\,dp_x \qquad (4)$$

and the distribution function $f(x,p_x;s)$ satisfies the Vlasov equation

$$\frac{\partial f}{\partial s}+\{H,f\}=0. \qquad (5)$$

Eqs.(1)~(5) are closed, and space charge problems are completely described by solving these equations in a self-consistent way. Following the usual procedure to solve the Vlasov-Poisson equation system, we separate V(x;s) and $f(x,p_x;s)$ into two parts, the stationary part and perturbing part, *i.e.*

$$\begin{cases} V(x;s)=V_0(x)+\delta V(x;s) & (6\,a)\\ f(x,p_x;s)=f_0(x,p_x)+\delta f(x,p_x;s). & (6\,b)\end{cases}$$

Accroding to eq.(6.b), we have for the number density

$$n(x;s)=n_0(x)+\delta n(x;s). \qquad (6\,c)$$

Substituting Eq.(2) together with Eq.(6) into Eq.(5) and neglecting the second order perturbing term, we obtain the linearized equations as follows:

$$p_x\frac{\partial f_0}{\partial x}-\left[\left(\frac{\omega_0}{u}\right)^2 x-\frac{q}{m_0 u^2}E_0(x)\right]\frac{\partial f_0}{\partial p_x}=0 \qquad (7\,a)$$

and

$$\frac{\partial \delta f}{\partial s}+p_x\frac{\partial \delta f}{\partial x}-\left[\left(\frac{\omega_0}{u}\right)^2 x-\frac{q}{m_0 u^2}E_0(x)\right]\frac{\partial \delta f}{\partial p_x}=-\frac{q}{m_0 u^2}\delta E(x;s)\frac{\partial f_0}{\partial p_x} \qquad (7\,b)$$

where we have put $E(x;s)=E_0(x;s)+\delta E(x;s)$.

Stationary space charge potential

To apply the linearized Vlasov approach to the closed equation system given in the previous section, the first issue is to get the stationary state distribution function $f_0(x,p_x;s)$. Neglecting the perturbing parts, the stationary state is described by the Hamiltonian

$$H_0=\frac{1}{2}p_x^{\,2}+\frac{1}{2}\left(\frac{\omega_0}{u}\right)^2 x^2+\frac{q}{m_0 u^2}V_0(x), \qquad (8)$$

and it is obvious that any function of H_0 is a solution of Eq.(7a), *i.e.* $f_0(x,p_x)=f_0(H_0)$. Accrdingly, we have from Eq.(3)

$$\frac{d^2 V_0}{dx^2}=-\frac{q}{\varepsilon_0}\int f_0\left(\frac{p_x^{\,2}}{2}+\frac{\kappa_0^{\,2}x^2}{2}+\frac{q}{m_0 u^2}V_0\right)dp_x \qquad (9)$$

where $\kappa_0=\omega_0/u$.

The stationary state potential is obtained by solving Eq.(9) while the form of the function f_0 depends on our choice. In the stationary state, the potential $V_0(x)$ should be an even function with respect to x, and the constant term can be neglected. Then, we can assume the form of $V_0(x)$ as

$$V_0(x)=-\frac{m_0 u^2}{q}\left[\frac{\kappa_0^{\,2}}{2}x^2-\sum_{n=1}^{\infty}w_n x^{2n}\right] \qquad (10)$$

Substitution of Eq.(10) into Eq.(9) leads to

$$\kappa_0^{\,2}-2\sum_{n=0}^{\infty}(n+1)(2n+1)w_{n+1}x^{2n}=\frac{q^2}{\varepsilon_0 m_0 u^2}n_0(x) \qquad (11)$$

where the number density is given by

$$n_0(x)=\sum_{n=0}^{\infty}v_n x^{2n}=\int f_0\left(\frac{p_x^{\,2}}{2}+\sum_{n=1}^{\infty}w_n x^{2n}\right)dp_x. \qquad (12)$$

By comparing the coefficients of the same order powers in Eq.(11) together with Eq.(12), we obtain

$$\begin{cases}\kappa_0^{\,2}-2w_1=\dfrac{q^2}{\varepsilon_0 m_0 u^2}\cdot v_0\equiv\kappa_p^{\,2} & (\text{for } n=0)\ (13.\,a)\\ 2(n+1)(2n+1)w_{n+1}=-\kappa_p^{\,2}\cdot\dfrac{v_n}{v_0} & (\text{for } n\neq 0)\ (13.\,b)\end{cases}$$

where we have defined the plasma frequency as $\omega_p^2=q^2v_0/\varepsilon_0 m_0$, and $\kappa_p=\omega_p/u$.

Now, we introduce the waterbag distribution defined by

$$f_0(H_0)=\frac{N}{2C}\left[1+sgn\left(\frac{E_{WB}}{2}-\frac{p_x^{\,2}}{2}-\sum_{n=1}^{\infty}w_n x^2\right)\right] \qquad (14)$$

where N and C are, respectively, the total particle number and normalization constant given by

$$C=\iint\frac{1}{2}\left|1+sgn\left(\frac{E_{WB}}{2}-\frac{p_x^{\,2}}{2}-\sum_{n=1}^{\infty}w_n x^{2n}\right)\right|dx\,dp_x. \qquad (15)$$

and E_{WB} is a constant closely related to the beam emittance.

From Eqs.(12) and (14), we have

$$\sum_{n=0}^{\infty} v_n x^{2n} = \frac{2N\sqrt{E_{WB}}}{C}\left[1 - \sum_{n=1}^{\infty} \frac{(2n-3)!!}{n!\,E_{WB}^{\,n}}\left(\sum_{m=1}^{\infty} w_m x^{2m}\right)^n\right]$$

and, therefore,

$$\begin{cases} v_0 = \dfrac{2N\sqrt{E_{WB}}}{C} \\ v_1 = -v_0 \cdot \dfrac{w_1}{E_{WB}} \\ v_2 = -v_0 \cdot \left[\dfrac{w_2}{E_{WB}} + \dfrac{1}{2}\left(\dfrac{w_1}{E_{WB}}\right)^2\right] \\ v_3 = -v_0 \cdot \left[\dfrac{w_3}{E_{WB}} + \dfrac{w_1 w_2}{E_{WB}^{\,2}} + \dfrac{1}{2}\left(\dfrac{w_1}{E_{WB}}\right)^3\right] \\ \qquad \vdots \end{cases}$$

Using the above equations and Eqs.(13), the coefficients w_n can be evaluated as follows:

$$\begin{cases} w_1 = \dfrac{\kappa_s^{\,2}}{2} \\ w_2 = \dfrac{1}{24\,E_{WB}}(\kappa_s \kappa_p)^2 \\ w_3 = \dfrac{1}{240\,E_{WB}^{\,2}}(\kappa_s \kappa_p)^2\left(\kappa_s^{\,2} + \dfrac{\kappa_p^{\,2}}{3}\right) \\ w_4 = \dfrac{1}{896\,E_{WB}^{\,3}}(\kappa_s \kappa_p)^2\left[\kappa_s^{\,4} + \dfrac{6}{15}(\kappa_s \kappa_p)^2 + \dfrac{\kappa_p^{\,4}}{45}\right] \\ \qquad \vdots \end{cases}$$

where $\kappa_s^2 = \kappa_0^2 - \kappa_p^2$.

Then, the stationary state Hamiltonian is

$$H_0 = \frac{1}{2}p_x^{\,2} + \frac{1}{2}\kappa_s^{\,2}x^2 + \frac{(\kappa_s\kappa_p)^2}{24}\left[\frac{x^4}{E_{WB}} + \frac{1}{10}\left(\kappa_s^{\,2} + \frac{\kappa_p^{\,2}}{3}\right)\frac{x^6}{E_{WB}^{\,2}} + \frac{3}{112}\left(\kappa_s^{\,4} + \frac{6}{15}\kappa_s^{\,2}\kappa_p^{\,2} + \frac{\kappa_p^{\,4}}{45}\right)\frac{x^8}{E_{WB}^{\,3}} + \cdots\right]. \qquad (16)$$

Note that the stationary state includes the infinite chain of the non-linear terms. For low-intensity beams, the higher order powers in Eq.(16) is small and the linear-force term is dominant. In the case of intense beams, the higher order terms make dominant contribution to the stationary state, and this indicates that the stationary distribution in real space becomes more and more uniform with increasing beam intensity. It is known as the homogenization effect. The external potential is canceled by the quadratic term in the space charge potential and, as a result, particles near the beam axis is almost in force-free state[1].

Stability of beam plasma oscillations

In this section, we consider

$$H_0 \to H_1 = \frac{1}{2}p_x^{\,2} + \frac{1}{2}\kappa_s^{\,2}x^2, \qquad (17)$$

as the stationary state Hamilotian

To achieve the self-consistent stability analysis, we must, of course, start with the Hamiltonian Eq.(16). But it is hopeless to solve analytically, so let us take mathematical simplicity rather than the perfect self-consistency. This simplification is physically valid, at least, for low-intensity beams, and it would be usefull to investigate the dynamical beam behavior. In this case, the perturbing Hamiltonian is

$$H = H_1 + \frac{q}{m_0 u^2}\delta V(x;s), \qquad (18)$$

and the Poisson's equation is given by

$$\frac{\partial^2 \delta V(x;s)}{\partial x^2} = -\frac{\partial \delta E(x;s)}{\partial x} = -\frac{q}{\varepsilon_0}\int \delta f(x, p_x; s)\,dp_x. \qquad (19)$$

When we write the beam emittance as ε_1, the stationary distribution function becomes

$$f_0(I_1) = \frac{N}{2\pi\varepsilon_1}\left[1 + sgn\left(\frac{\varepsilon_1}{2} - I_1\right)\right] \qquad (20)$$

where we have introduced the action-angle variables (I_1, θ_1) defined by

$$x = \sqrt{\frac{2I_1}{\kappa_s}}\cos\theta_1 \quad and \quad p_x = -\sqrt{2\kappa_s I_1}\sin\theta_1.$$

The linearzied Vlasov equation (7b) is rewritten with the action-angle variables as

$$\frac{\partial \delta f}{\partial s} + \kappa_s\frac{\partial \delta f}{\partial \theta_1} = \frac{q}{m_0 u^2}\sqrt{\frac{2I_1}{\kappa_s}}\,\delta E(\theta_1, I_1; s)\sin\theta_1\frac{df_0(I_1)}{dI_1}. \qquad (21)$$

Now, we expand δf into Fourier series as

$$\delta f(\theta_1, I_1; s) = \exp(-i\kappa s)\sum_{n=-\infty}^{\infty}\delta\tilde f_n(I_1)\exp(in\theta_1).$$

Substituting this and Eq.(20) into Eq.(21) and integrating with respect to θ_1, we obtain

$$2\pi i(\kappa - n\kappa_s)\delta\tilde f_n(I_1) = \frac{Nq}{\pi m_0 u^2\sqrt{\kappa_s\varepsilon_1}}\delta\left(\frac{\varepsilon_1}{2} - I_1\right) \times \int_0^{2\pi}\delta\tilde E(\theta_1, I_1 = \varepsilon_1/2)\sin\theta_1\exp(-in\theta_1)\,d\theta_1 \qquad (22)$$

where $\delta(z)$ is the Dirac's delta function and

$$\delta E(\theta_1, I_1; s) = \delta\tilde E(\theta_1, I_1)\exp(-i\kappa s).$$

Eq.(22) indicates that we can put

$$\delta\tilde f_n(I_1) = \tilde f_n \cdot \delta\left(\frac{\varepsilon_1}{2} - I_1\right) \qquad (23)$$

which means that the perturbation occurs only at the beam edge.

Noting that the solution of Eq.(19) is given by

$$\delta\tilde E = \frac{q}{2\varepsilon_0}\sum_{n=-\infty}^{\infty}\iint sgn(\sqrt{I_1}\cos\theta_1 - \sqrt{I'_1}\cos\theta'_1)\,\delta\tilde f_n(I'_1)\,e^{in\theta'_1}\,d\theta'_1\,dI'_1,$$

Eq.(22) can be rewritten as

$$2\pi i(\kappa - n\kappa_s)\tilde f_n = \frac{i\kappa_p^{\,2}}{4\kappa_s}\sum_{m=-\infty}^{\infty}F_{nm}\tilde f_m \qquad (24)$$

with the use of Eq.(23). Here, F_{nm} can be evaluated as

$$F_{nm} = -i\int_0^{2\pi}d\theta_1\cdot\sin\theta_1\,e^{-in\theta_1}\cdot\int_0^{2\pi}d\theta'_1\cdot sgn(\cos\theta_1 - \cos\theta'_1)\,e^{im\theta'_1}$$
$$= \begin{cases} -\dfrac{32n}{\left[(m-n)^2 - 1\right]\left[(m+n)^2 - 1\right]}, & for\quad m+n = even \\ 0, & for\quad m+n = odd \end{cases}$$

Accordingly, Eq.(24) becomes

$$\sum_{m=-\infty}^{\infty}F_{nm}\cdot\tilde f_m = \kappa\cdot\tilde f_n \qquad (25)$$

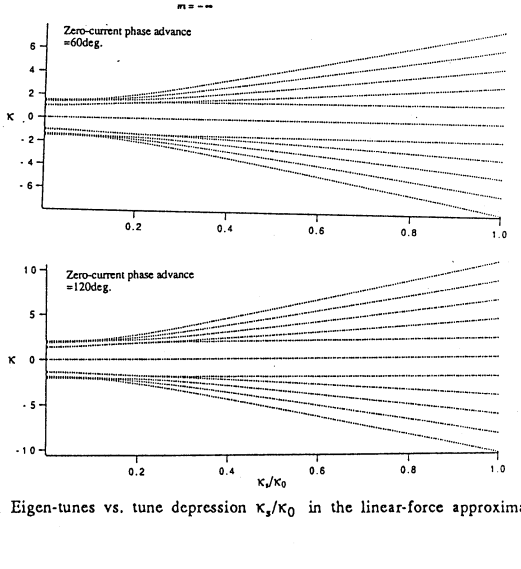

Fig.1 Eigen-tunes vs. tune depression κ_s/κ_0 in the linear-force approximation

where the matrix element is given by

$$F_{nm} = \kappa_s\left[m \cdot \delta_{nm} + \frac{1}{8\pi}\left(\frac{\kappa_p}{\kappa_s}\right)^2 \cdot F_{nm}\right]$$

and δ_{nm} is the Kronecker's delta.

Eq.(25) is an eigen-value equation to determine the frequencies of the beam plasma oscillation modes. Fig.1 shows the real part of the eigen-tune κ evaluated from Eq.(25) in the case where the first five modes are taken into consideration. The eigen-tune κ is always real, and this means that there exists no instability.

A non-linear force approximation

As previously mentioned, the non-linear terms in the stationary potential become more dominant in higher intensity beams. Therefore, in this section, let us investigate the stationary state described by the Hamiltonian

$$H_0 \rightarrow H_2 = \frac{1}{2} p_x^2 + \frac{(\kappa_s \kappa_p)^2}{24 E_{WB}} x^4. \tag{26}$$

While, to achieve higher self-consistency in the stability analysis for intense beams, we must take some non-linear-force terms simultaneously into consideration, the simple cubic-force approximation may be suitable for our purpose because it allows us fully analytical description of the dynamical beam behavior. Additionally, we can investigate whether the non-linearity of the stationary potential will cause some essential change in the results, compared with those obtained from the linear approximation. In this approximation, the new emittance ε_2 can be defined as

$$\varepsilon_2 = \frac{1}{\pi} \oint p_x(x;\, H_2 = \frac{E_{WB}}{2})\, dx = \frac{E_{WB}}{\eta\sqrt{\kappa_s \kappa_p}} \tag{27}$$

where

$$\eta = \frac{3^{3/4}\pi}{8 \cdot K(1/\sqrt{2})} \cong 0.483\,,$$

and K(k) is the complete elliptic integral of the first kind:

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}.$$

From Eqs.(26) and (27), the total Hamiltonian considered here is given by

$$H = H_2 + \frac{q}{m_0 u^2}\delta V(x;\, s) \tag{28}$$

where

$$H_2 = \frac{1}{2} p_x^2 + \frac{(\kappa_s \kappa_p)^{3/2}}{24\eta\varepsilon_2} x^4.$$

Introducing the action-variable defined by

$$I_2 = \frac{1}{2\pi} \oint p_x(x;\, H_2)\, dx = \left[\frac{\varepsilon_2^{1/3}}{2^{1/3} \cdot \eta\sqrt{\kappa_s \kappa_p}} \cdot H_2\right]^{3/4},$$

H_2 is rewritten as

$$H_2 = \frac{2^{1/3}\eta\sqrt{\kappa_s \kappa_p}}{\varepsilon_2^{1/3}} \cdot I_2^{4/3}$$

and the action-variable is derived from the Hamiltonian equation of motion, *i.e.*

$$\frac{d\theta_2}{ds} = \frac{\partial H_2}{\partial I_2} = \frac{2^{7/3}\eta\sqrt{\kappa_s \kappa_p}}{3}\left(\frac{I_2}{\varepsilon_2}\right)^{1/3} \equiv k_c(I_2).$$

Using these canonical variables, the stationary waterbag distribution is given by

$$g_0(I_2) = \frac{N}{2\pi\varepsilon_2}\left[1 + sgn\left(\frac{\varepsilon_2}{2} - I_2\right)\right]$$

Applying the same technique as presented in the previous section to the Hamiltonian (28) together with the total distribution function $g(I_2,\theta_2;s)=g_0(I_2)+\delta g(I_2,\theta_2;s)$, we obtain[1]

$$\begin{cases} 2\pi i(\kappa - n\kappa_c)\, g_n = \dfrac{q}{m_0 u^2}\dfrac{3N}{\pi\sqrt{8\eta\varepsilon_2}\,(\kappa_s \kappa_p)^{1/4}} \\ \qquad \times \displaystyle\int_0^{2\pi} \delta E(\theta_2, I_2 = \frac{\varepsilon_2}{2}) \cdot DN(\sigma\theta_2)\, SN(\sigma\theta_2)\, e^{-in\theta} \cdot d\theta_2 & (29.a) \\ \delta E(\theta_2, I_2 = \dfrac{\varepsilon_2}{2}) = \dfrac{q}{2\varepsilon_0} \displaystyle\sum_{n=-\infty}^{\infty} g_n \int sgn(CN(\sigma\theta_2) - CN(\sigma\theta_2'))\, e^{in\theta'} \cdot d\theta_2' & (29.b) \end{cases}$$

where SN(x), CN(x) and DN(x) are the Jacobian elliptic functions,

$$\kappa_c = k_c(I_2 = \frac{\varepsilon_2}{2}) = \frac{4\eta\sqrt{\kappa_s \kappa_p}}{3} \cong 0.64\sqrt{\kappa_s \kappa_p}\,,$$

$$\sigma = \frac{2K(1/\sqrt{2})}{\pi} = \frac{3^{3/4}}{4\eta} \cong 1.18,$$

and we have put

$$\begin{cases} \delta g(\theta_2, I_2;\, s) = \delta\left(\dfrac{\varepsilon_2}{2} - I_2\right)\exp(-i\kappa s) \displaystyle\sum_{n=-\infty}^{\infty} g_n \cdot \exp(in\theta_2) \\ \delta E(\theta_2, I_2;\, s) = \delta E(\theta_2, I_2)\exp(-i\kappa s) \end{cases}$$

Eqs.(29) lead to

$$2\pi i(\kappa - n\kappa_c)\, g_n = i\frac{3\kappa_p^2}{16\eta\sqrt{\kappa_s \kappa_p}} \sum_{m=-\infty}^{\infty} G_{nm}\, g_m \tag{30}$$

where the matrix element G_{nm} can be evaluated as

$$G_{nm} = -i\sqrt{2}\int_0^{2\pi} d\theta_2 \cdot DN(\sigma\theta_2)\, SN(\sigma\theta_2)\, e^{-in\theta} \cdot \times \int_0^{2\pi} d\theta_2' \cdot sgn(CN(\sigma\theta_2) - CN(\sigma\theta_2'))\, e^{im\theta'}$$

$$= \begin{cases} -\dfrac{4096\sqrt{3}\,\eta^2}{9} \displaystyle\sum_{k=0}^{\infty} \frac{(2k+1)^2 n}{\left[(m+n)^2 - (2k+1)^2\right]\left[(m-n)^2 - (2k+1)^2\right]} \cdot A_k & \text{, for } m+n = \textit{even} \\ 0 & \text{, for } m+n = \textit{odd} \end{cases}$$

and $A_n=e^{-(n+0.5)\pi}/[1+e^{-(2n+1)\pi}]$.

Thus, from Eq.(30), we have

$$\sum_{m=0}^{\infty} G_{nm} \cdot g_m = \kappa \cdot g_n \tag{31}$$

where

$$G_{nm} = \kappa_c\left[m \cdot \delta_{nm} + \frac{1}{8\pi}\left(\frac{\kappa_p}{\kappa_c}\right)^2 \cdot G_{nm}\right]$$

The eigen-tunes evaluated from Eq.(31) are, again, always real at any value of the tune depression κ_s/κ_0(see Fig.2).

Eqs.(25) and (31) are very similar to each other, and there is no essential change. In fact, the both equations have only the real eigen-values. While the cubic-force approximation introduced here is rather crude, it is expected that 1D waterbag-type beams are essentially stable even in the more self-consistent analysis.

Reference

1). H. Okamoto, to be submitted elsewhere.

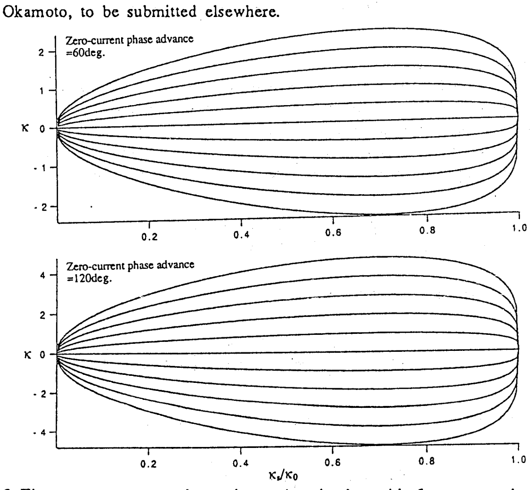

Fig.2 Eigen-tunes vs. tune depression κ_s/κ_0 in the cubic-force approximation